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formal überarbeitete Version der Originalveröffentlichung in:
formally revised edition of the original source in:
Billich, Martin [Hrsg.]: Teaching Mathematics III. Innovation, new trends, research. Ruzomberok, Slovakia : VERBUM - Catholic University in Ruzomberok Press 2012, S. 5-13

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https://doi.org/10.25656/01:7186

Basic trigonometric formulas in an inquiring approach

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Learning without thinking is labor lost; thinking without learning is perilous.
Confucius: Confucian analects
That is what learning is. You suddenly understand something you've understood all your life, but in a new way.
Doris Lessing

Abstract. The main aim of this paper is to present an inquiry based professional development activity about the teaching of basic trigonometric relations and some conclusions about possible implementations in the framework of regular school lessons. The activity itself was designed to understand basic facts about the structure and the construction of standard curricula parts and in the same time to achieve a higher consciousness level in choosing teaching attitudes.

Keywords: basic trigonometric formulas, inquiry based learning
Mathematical Subject Classification: 97G60

Introduction

In the recent educational trend the introduction of inquiry based learning into day to day teaching practice has become an important goal ([5], [3]). However the real world approach based on modelling and inquiry steps was traditionally an organic part of mathematics teaching (not as a pedagogical method, but as a strategy for understanding [2]) the politicians and decision-making system needs new proofs for the efficiency and reliability of this approach ([6]). But to obtain demonstrative arguments a more organic structure is needed in the construction of curricula. This structure has to deal not only with pieces of content as independent entities but it has to deal with the teaching method itself, the activities students have to perform in order to achieve a deep understanding of mathematics as a science, as a human activity, as a useful viewpoint in handling various situations, a viewpoint that ultimately models human behavior. On the other hand teachers needs new materials for the implementation of IBL and moreover they need transformative training in order to get confidence in using IBL.

As a part of the FP7 project PRIMAS[2] the Babeş-Bolyai University organized several training courses on the practical aspects of inquiry based teaching of mathematics. This paper is a report of one session which was focused on the teaching of trigonometric formulas. The main aim of this session was double: to get a deep insight into classroom processes, learning difficulties, teacher attitudes, student reactions which can appear during the teaching of trigonometric formulas and on the other hand to develop an organic structure, suitable for IBL activities at classroom and at professional development level. According to these targets the session was divided into two major parts. In the first step of part I.(about 1,5 working hours) the participating teachers worked in small groups and each small group had to construct a detailed teaching approach with problems and proofs. To avoid similar or identical approaches at the beginning there was a brief brainstorming to emphasize all possible ideas and then each group had to work on a specific idea he has chosen. As a second step (about 1,5 hours) there was a debate on the constructed proofs, on the different approaches from the point of view of teaching practice. In the second part the teachers had to construct a new curriculum structure for this content unit based on IBL activities and real world application. For this part a consultation with several professionals (topographer, astronomer, web designer, architect) was used and after these consultations the teachers had to prepare a common structure. This report is focused on the first part, where teachers had to develop and to reflect on possible proving strategies for the basic formulas.

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Proofs and strategies

Problem 1. *Construct a strategy to calculate* $\sin(x+y), \cos(x+y)$ *and* $\text{tg}(x+y)$ *if you know* $\sin(x), \cos(x)$, $\text{tg}(x)$ *and* $\sin(y), \cos(y), \text{tg}(y)$.

After several comments the participants agreed that only geometric proofs will be considered because only this can guarantee a strong connection to students' former knowledge (in Romania trigonometry is taught in the 9^{th} grade, for 15 years old students). Some other approaches can be found in [10], these can be used to connect trigonometry and vector geometry, or complex number, but first the basic formulas need to be derived by elementary arguments. This shows that for a succesfull calculation (or proof) we need to construct a figure where we have the angles x, y and $x+y$ embedded into right angled triangles. After a short brainstorming the configurations from Figure 1. were found.

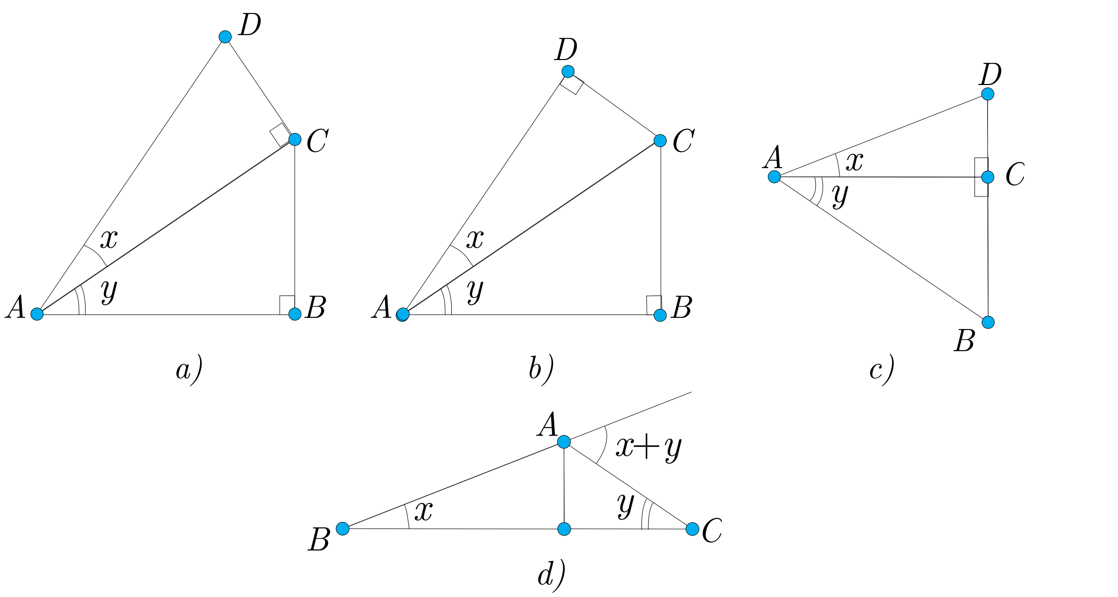

Figure 1: Possibilities to illustrate x, y and $x+y$, where x, y are in right angled triangles

Each group had to choose a configuration and to construct a proof and to analyze possible other proofs based on the chosen configuration.

Configuration a) We need to embed the angle $x+y$ into a right angled triangle. For this we have several possibilities (in fact there are infinitely many, but only a few are connected to the already constructed configuration). Hence we have to deal with the possibilities illustrated on Figure 2 (we omitted the case when a perpendicular to AD is drawn through C).

Case 1. We use the first configuration of Figure 2. In order to be able to calculate the lengths of segments in the figure we need have a reference length, which can be any one of the existing segments. For the simplicity we choose $AD = 1$. The strategy for calculation is very simple. We calculate the length of each segment using x, y and AD. We can fix $AD = 1$. Hence $DC = \sin x$, $AC = \cos x, BC = \cos x \sin y$ and $AB = \cos x \cos y$. To calculate DE we construct the orthogonal projection of C to DE. With this construction we have $m(\widehat{CDF}) = y$, so $DF = \sin x \cos y$. On the other hand

$$\sin(x+y) = \frac{DE}{1} = DE = DF + FE = DF + BC = \sin x \cos y + \cos x \sin y.$$

Using a similar argument we obtain

$$\cos(x+y) = \frac{AE}{1} = AE = AB - BE = AB - FC = \cos x \cos y - \sin x \sin y$$

and

$$\text{tg}(x+y) = \frac{DE}{AE} = \frac{DF + FE}{AB - FC} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\text{tg}x + \text{tg}y}{1 - \text{tg}x\text{tg}y}.$$

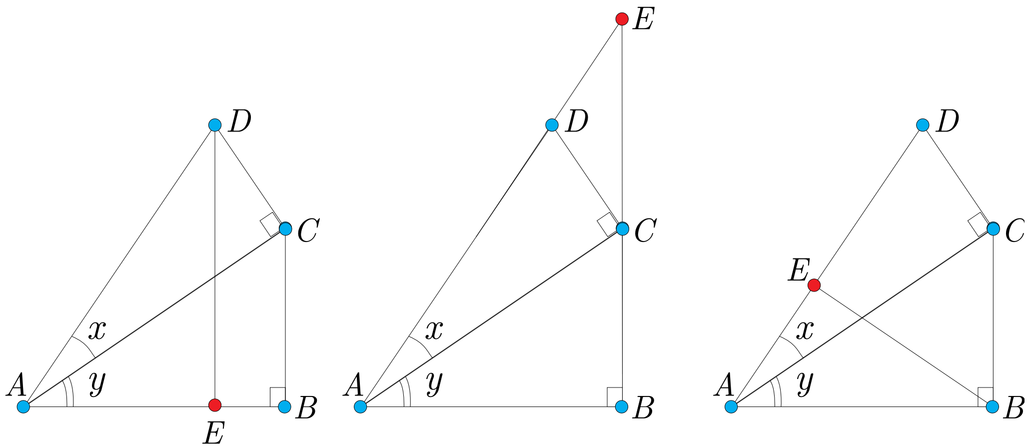

Figure 2: Possibilities to embed $x + y$ into a right angled triangle

Figure 3 illustrates the necessary constructions for the second and the third configuration of Figure 2. For completeness we reproduce the proofs for these figures too.

Remark 1. *This proof can be found in many textbooks and online resources. See for example [1], [10].*

Case 2. On the second configuration of Figure 3 we fix $AD = 1$ and we have $FC = \sin x \cos y$, $DF = \sin x \sin y$, $DE = \dfrac{\sin x \sin y}{\cos(x+y)}$ and $EF = \sin x \sin y \operatorname{tg}(x+y)$. Hence

$$\sin(x+y) = \frac{BE}{AE} = \frac{\cos x \sin y + \sin x \cos y + \sin x \sin y \operatorname{tg}(x+y)}{1 + \frac{\sin x \sin y}{\cos(x+y)}}.$$

From this equality we obtain (after rearrangement)

$$\sin(x+y) = \sin x \cos y + \sin y \cos x. \tag{1}$$

The formulas for $\cos(x+y)$ and $\operatorname{tg}(x+y)$ can be obtained in a similar manner.

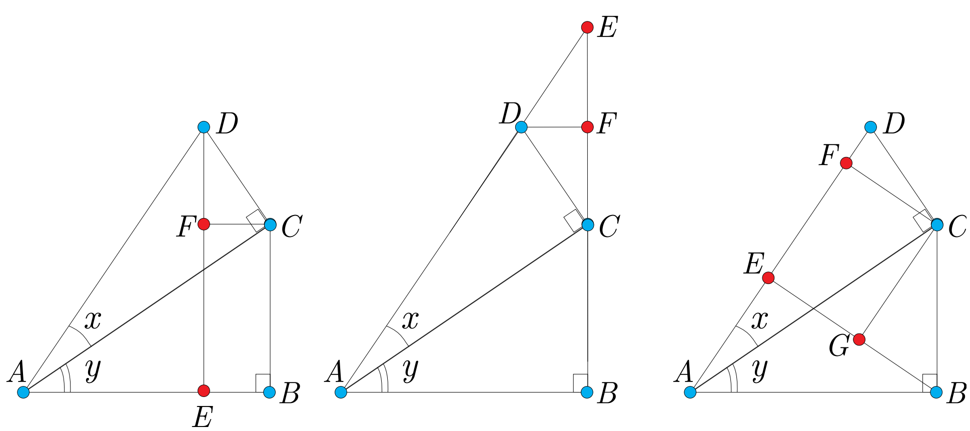

Figure 3: Auxiliary lines and calculation strategies I.

Case 3. We use the third configuration of Figure 3 and we consider as in the previous cases $AD = 1$. In this construction we have $m(\widehat{FCD}) = x$ and $m(\widehat{GBC}) = x + y$, thus

$$\sin(x+y) = \frac{GC}{BC} = \frac{EF}{BC} = \frac{AD - AE - FD}{BC}.$$

On the other hand $AE = AB\cos(x+y) = \cos x \cos y \cos(x+y)$, $FD = DC\sin x = \sin^2 x$ and $BC = \cos x \sin y$, so we obtain

$$\sin(x+y)\sin y + \cos(x+y)\cos y = \cos x \tag{2}$$

In order to obtain another relation between $\sin(x+y)$ and $\cos(x+y)$ we write $EB = AB\sin(x+y) = EG + GB = FC + BC\cos(x+y)$ so

$$\sin(x+y)\cos y - \sin y\cos(x+y) = \sin x. \tag{3}$$

Multiplying both sides of (2) with $\sin y$, both sides of (3) with $\cos y$ and adding the obtained relations we deduce 1 and from this we can obtain the corresponding formula for cosine and tangent.

Configuration b) Using configuration $b)$ for the embedding of the angle $x+y$ into a right angled triangle we have the possibilities illustrated in Figure 4 (for each case we illustrated the auxiliary construction too). According to these constructions we have to analyze the following three cases.

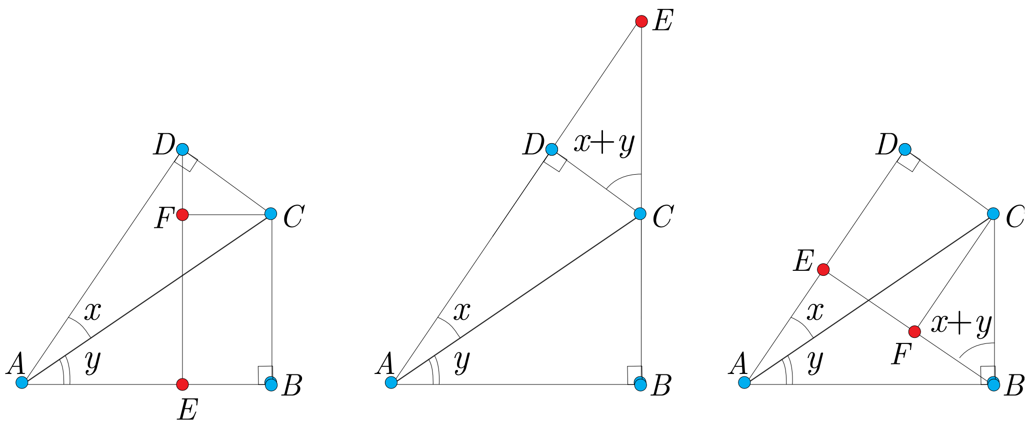

Figure 4: Auxiliary lines and calculation strategies II.

Case 1. We use the first construction on Figure 4 and we consider $AC = 1$. $m(\widehat{FDC}) = x+y$, so $FD = DC\cos(x+y) = \sin x\cos(x+y)$, $FE = BC = \sin y$ and hence

$$\sin(x+y) = \frac{DE}{AD} = \frac{\sin x\cos(x+y) + \sin y}{\cos x}.$$

From this relation we obtain

$$\sin(x+y)\cos x - \cos(x+y)\sin x = \sin y \tag{4}$$

On the other hand $AB = AE + EB = AE + FC$, hence

$$\sin(x+y)\sin x + \cos(x+y)\cos x = \cos y \tag{5}$$

From relation (4) and (5) we obtain the well known formulas for $\sin(x+y)$, $\cos(x+y)$.

Remark 2. *It is possible to use a slightly different argument in this case. In the triangle ADE we have* $\sin(x+y) = \frac{DE}{AD} = \frac{BC+DF}{AD} = \frac{\sin x\cos(x+y)+\sin y}{\cos x}$. *From this relation we have*

$$\cos^2(x+y) = 1 - \sin^2(x+y) = 1 - \frac{\sin^2 x\cos^2(x+y) + \sin^2 y + 2\sin x\sin y\cos(x+y)}{\cos^2 x,}$$

hence we obtain a quadratic equation for $\cos(x+y)$:

$$\cos^2(x+y) + 2\sin x\sin y\cos(x+y) + \sin^2 y - \cos^2 x = 0.$$

The discriminant of this equation is

$$\Delta = 4\sin^2 x \sin^2 y - 4\sin^2 x + 4\cos^2 y = 4\cos^2 x \cos^2 y,$$

thus

$$\cos(x+y) = -\sin x \sin y \pm \cos x \cos y.$$

In the last relation the last term could not have negative sign because $\cos(x+y)$ *must be positive, hence we obtain the formula for* $\cos(x+y)$ *and from this we can obtain the formula for* $\sin(x+y)$ *and* $\mathrm{tg}(x+y)$.

Case 2. Suppose $AC = 1$. $BC = \sin y$ and $EB = \cos y \mathrm{tg}(x+y)$ and from the triangle DEC we have $EC = \sin x \frac{1}{\cos(x+y)}$. From these relation we have

$$\sin y = BC = EB - EC = \mathrm{tg}(x+y)\cos y - \sin x \frac{1}{\cos(x+y)},$$

or

$$\sin x + \cos(x+y)\sin y = \cos y \sin(x+y).$$

Squaring both sides and using $\sin^2(x+y) + \cos^2(x+y) = 1$ we obtain the same quadratic equations for $\cos(x+y)$ as in the previous remark, hence the rest of the proof is the same.

Case 3. As in the previous cases we suppose $AC = 1$. From $AD = AE + ED = AE + CF$ and $DC = EF = EB - FB$ we obtain the relations (2) and (3), hence the rest of the proof is similar to the third case of configuration a).

Configuration c) Working with this configuration allows to work with other tools. If we calculate the aria of the triangle ABD in two different ways, we obtain a formula for $\sin(x+y)$. Suppose $AC = 1$. On one hand we have

$$\sigma[ABD] = \frac{AC \cdot CD}{2} + \frac{AC \cdot BC}{2} = \frac{\mathrm{tg}x + \mathrm{tg}y}{2}.$$

On the other hand

$$\sigma[ABD] = \frac{AD \cdot AB \cdot \sin(x+y)}{2}.$$

From these relations we obtain

$$\sin(x+y) = \frac{DC + BC}{AB \cdot AD} = \cos x \cos y(\mathrm{tg}x + \mathrm{tg}y) = \cos y \sin x + \cos x \sin y.$$

Remark 3. *The use of the are can be avoided if we construct the perpendicular* BE *from* B *to* AD. *In this case* $BF = \mathrm{tg}y \cos x$ *and* $EF = \sin x$, *where* F *is the projection of* C *to* EB.

Configuration d) For the last configuration we can use several auxiliary constructions (if we want to embed the angle $x+y$ into a right angled triangle) or we can use an argument based on the area of the triangle (for this last argument see also [8]). We analyze two auxiliary constructions. These are illustrated in Figure 5.

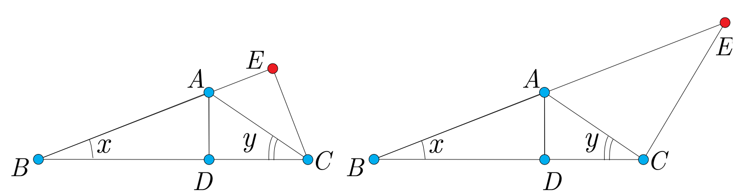

Figure 5: Auxiliary lines and calculation strategies III.

Case 1. Suppose $AD = 1$. With this assumption we have $BD = \frac{1}{\mathrm{tg}x}$, $DC = \frac{1}{\mathrm{tg}y}$, $AC = \frac{1}{\sin y}$ and $EC = \left(\frac{1}{\mathrm{tg}x} + \frac{1}{\mathrm{tg}y}\right)\sin x$. From the triangle EAC we obtain

$$\sin(x+y) = \frac{EC}{AC} = \sin x \sin y \left(\frac{1}{\mathrm{tg}x} + \frac{1}{\mathrm{tg}y}\right) = \sin x \cos y + \sin y \cos x.$$

In a similar way we can deduce the formulas for $\mathrm{tg}(x+y)$, $\cos(x+y)$.

Remark 4. *This proof appears in [13] and [14].*

Case 2. In this case we denote by a and b the length of CE respectively AE. From the aria of triangle BCE we obtain

$$a\frac{1}{\sin y}+\frac{1}{\mathrm{tg}x}+\frac{1}{\mathrm{tg}y}=\left(\frac{1}{\mathrm{tg}x}+\frac{1}{\mathrm{tg}y}\right)\left(\frac{1}{\sin x}+b\right)\sin x,$$

so

$$\frac{a}{b}=\sin x\sin y\left(\frac{1}{\mathrm{tg}x}+\frac{1}{\mathrm{tg}y}\right).$$

On the other from the triangle ACE we have $\sin(x+y)=\frac{a}{b}$, hence we obtain the desired relation.

Remark 5. *Using more auxiliary constructions and other argument we can obtain also some other proofs. As an example we sketch a proof based on Ptolemy's first theorem (for more delightful trigonometric ideas see [9]). In the configuration c) the quadrilateral $ABCD$ is inscribed in the circle with diameter AC, hence due to Ptolemy's first relation we have*

$$AC\cdot BD=AD\cdot BC+AB\cdot CD.$$

If we suppose $AC=1$ and we use the relations from case 1/a) we obtain

$$BD=\sin x\cos y+\sin y\cos x.$$

On the other hand $BD=\sin(x+y)$ (due to the sine law or using an auxiliary construction), hence we obtain the required relation.

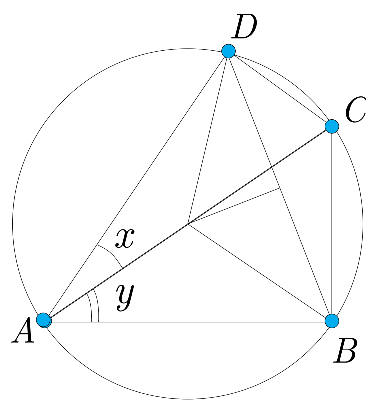

Figure 6: Auxiliary lines and calculation strategies IV.

Remark 6. *The formulas for $\sin(x-y)$, $\cos(x-y)$, $\mathrm{tg}(x-y)$ can be proved using similar arguments and figures. We propose to the reader as an exercise to analyze these geometrical proofs.*

Remark 7. *The main idea of the previous proofs also appears in [4].*

Conclusions

1. Almost all solutions appeared at the training session (produced by the groups in the initial step, or at the presentation of the possible solutions), the author of this paper only unified and structured them. The participants were highly surprised about the diversity of possible approaches in contrast to the usual treatment of the problem in textbooks and classrooms.
2. The participants realized that basically all possible ideas were good and all ideas led to a correct solution (with more or less effort). By analyzing these proofs they realized that in many textbooks the presented solution is the shortest or the simplest as reasoning. This aspect is very important and shows that usually without analyzing other possible approaches, proofs, the main properties of a proof remain hidden. In most cases they teach a possible proof, but they don't know that this proof is the optimal from some points of view.

3. The diversity of proofs shows that if the necessity of these formulas appears as a problem in contextual problem situations, students have real chance to construct a geometric proof which is strictly based on their previous knowledge. Moreover by giving them the chance to construct such a proof and analyze some alternative approaches they can realize how mathematics works, how mathematicians work. They can understand that some tools can be more effective than others in constructing a proof and doing mathematics means that we seek for an optimal solution/proof (which is not usually not unique because depends on the criteria we use).

4. Most of the proofs have one or two fundamental idea in the background, the rest are only technical details. The real understanding of mathematics starts with the understanding of these background ideas and the capability of making the connections between the simple and clear background ideas and formalism. All these proofs and ideas are useless in the existing Romanian curricula where first the trigonometric functions are introduced and the proofs are fitted to this approach (while this functional viewpoint is strange for most of the students). This shows that if we want to use IBL methods and activities, we need to completely rethink the structure of the content and in most cases we have to start with the problems (applications) that are at the end of the chapter in the present structure. The trigonometry represents an illustrative example in this sense. The geometrical applications are usually at the end of the chapter while fundamentally they motivate the existence of the whole chapter and they connect this chapter to the students' former knowledge.

5. The proofs presented here shows how inquiry based learning is working in an abstract framework and how simple ideas (the necessity of constructing the right angled triangles) combined with basic choices (the choice of configuration) lead to a correct proof. From this point of view the content itself in this activity is not as important as the processes the students/teachers are going through. This shift (from focusing on content to focusing on processes) is proper to inquiry based teaching however the importance of the content must be emphasized (by using an adequate context to introduce the problem and by giving other applications too).

Acknowledgements

This paper is based on the work within the project Primas. Coordination: University of Education, Freiburg. Partners: University of Genève, Freudenthal Institute, University of Nottingham, University of Jaen, Konstantin the Philosopher University in Nitra, University of Szeged, Cyprus University of Technology, University of Malta, Roskilde University, University of Manchester, Babeş-Bolyai University, Sør-Trøndelag University College. The author wishes to thank his students and colleagues attending the training course organized by the Babeş-Bolyai University in the framework of the FP7 project PRIMAS[3]. The author was partially supported also by the SimpleX Association from Miercurea Ciuc.

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