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MERIA - conflict lines: experience with two innovative teaching materials

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MERIA – Conflict Lines: Experience with Two Innovative Teaching Materials

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The design of inquiry-based tasks and problem situations for daily mathematics teaching is still a challenge. In this article, we study the implementation of two tasks as part of didactic scenarios for inquiry-based mathematics teaching, examining teachers' classroom orchestration supported by these scenarios. The context of the study is the Erasmus+ project MERIA – Mathematics Education: Relevant, Interesting and Applicable, which aims to encourage learning activities that are meaningful and inspiring for students by promoting the reinvention of target mathematical concepts. As innovative teaching materials for mathematics education in secondary schools, MERIA scenarios cover specific curriculum topics and were created based on two well-founded theories in mathematics education: realistic mathematics education and the theory of didactical situations. With the common name *Conflict Lines* (Conflict Lines – Introduction and Conflict Set – Parabola), the scenarios aim to support students' inquiry about sets in the plane that are *equidistant* from given geometrical figures: a perpendicular bisector as a line equidistant from two points, and a parabola as a curve equidistant from a point and a line. We examine the results from field trials in the classroom regarding students' formulation and validation of the new knowledge, and we describe the rich situations teachers may face that encourage them to proceed by building on students' work. This is a crucial and creative moment for the teacher, creating opportunities and moving between students' discoveries and the intended target knowledge.

Keywords: inquiry-based mathematics teaching, realistic mathematics education, teaching scenarios, theory of didactic situations

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MERIA – Razmejitvene črte: izkušnje z dvema inovativnima učnima gradivoma

Željka Milin Šipuš, Matija Bašić, Michiel Doorman, Eva Špalj in Sanja Antoliš

Oblikovanje preiskovalnih nalog in problemskih situacij pri dnevnem poučevanju matematike je še vedno izziv. V tem prispevku prikazujemo vključevanje dveh nalog v pouk matematike kot primera didaktičnih scenarijev preiskovalnega načina poučevanja matematike in analizo učiteljevih implementacij teh scenarijev v svojih razredih. Prispevek je nastal pod okriljem Erasmus+, projekta MERIA – *matematično izobraževanje: relevantno, zanimivo in uporabno*, katerega cilj je spodbujati učne situacije, ki imajo za učence pomen, so zanje motivirani, saj omogočajo samostojno odkrivanje izbranih matematičnih pojmov. Scenariji MERIA kot inovativno učno gradivo za matematično izobraževanje v srednjih šolah zajemajo izbrane vsebine učnega načrta in temeljijo na dveh dobro utemeljenih teorijah matematičnega izobraževanja: realistična matematika in teorija didaktičnih situacij. S splošnim imenom Razmejitvene črte (razmejitvene črte – uvod in razmejitvena množica – parabola) je namen scenarijev podpreti preučevanje študentov o množicah točk v ravnini, ki so enako oddaljeni od danih geometrijskih objektov: pravokotnica kot premica, ki je enako oddaljena od dveh točk, ter parabola kot krivulja, ki je enako oddaljena od točke in premice. V prispevku prikazujemo proces učenčevega oblikovanja znanja in učne situacije, ki za učitelja predstavljajo dobro izhodišče za nadgrajevanje učenčevega znanja. Prepoznavanje takih situacij je ključno, saj ustvarjajo prostor za nove priložnosti oz. premikanje od učenčevih odkritij do ciljnega znanja.

Ključne besede: raziskovalno poučevanje matematike, realistično poučevanje matematike, scenariji poučevanja, teorija didaktičnih situacij

Introduction

In many countries worldwide, today's curricula promote student-centred teaching approaches and, in the case of mathematics, students' reinvention of mathematics is at stake. Although these ideas are not new in mathematics teaching, their large-scale expansion coincides with the immense expansion of human knowledge in all fields of activity. Society's demands for skill development, including inquiry and problem-solving skills, critical thinking, reasoning and creativity, are certainly reasonable in a knowledge-based society, and are seen as the responsibility of education. Mathematics, especially, is perceived as the educational domain in which these so-called twenty-first century skills could be addressed. The Erasmus+ project MERIA[5] – Mathematics Education: Relevant, Interesting and Applicable, which is one of many educational projects worldwide, thus aims to promote inquiry-based mathematics teaching (IBMT) as one of the teaching approaches that provides opportunities to address these demands. IBMT supports students' own inquiry of unstructured problem situations in which they work similarly to researchers by posing questions, experimenting and hypothesising, validating and evaluating. These ideas emerged even earlier in science education (Artigue & Blomhøj, 2013), and many projects have been launched that support the development and implementation of inquiry-based science education (IBSE), such as projects promoting relevant school science education at the secondary level (e.g., Holbrook & Rannikmäe, 2014). Similar ideas were in fact suggested in the 1940s by the educational researcher John Dewey, who saw teaching as related to students' experiences and promoted focusing on activities in which students "learn by doing" (Winsløw, 2017). Nowadays, IBMT attracts the attention of education researchers worldwide, in particular by considering its relation to the already well-established problem-solving tradition and other theoretical frameworks in mathematics education programmes (Artigue & Blomhøj, 2013). The design of appropriate tasks or problem situations that have the potential to engage students in inquiry activities is a very important element in this process, since, as the evidence shows, such tasks with teachers' instructions are often missing (Bruder & Prescott, 2013).

Theoretical framework

In order to support the implementation of IBMT in secondary schools, the Erasmus+ project MERIA offered specially designed teaching scenarios as an innovative teaching materials project for IBMT, with tasks and problem

5 See more on the project https://meria-project.eu/.

situations that were selected or developed for the purpose. The design was supported by two well-founded theories in mathematics education: realistic mathematics education (RME) and the theory of didactic situations (TDS). Realistic mathematics education is based on the idea that students' experiences can be considered as a starting point for their own inquiry in mathematics (Freudenthal, 1991). It promotes the learning of mathematics as meaningful and related to different kinds of human activity, including pure mathematics itself, by using rich contexts that are "familiar for students and provide relevant and challenging elements that need to be organized or schematized mathematically so as to have the potential to evoke their (informal) knowledge" (Kieran et al., 2013, p. 53). The theory of didactic situations (Brousseau, 1997) assumes that students can construct new knowledge in an appropriately constructed *didactical milieu*, as a specially designed teaching environment that contains a mathematical problem and with which students interact autonomously, without the teacher's guidance. To this end, a milieu should have adidactical potential that enables students to work alone, and that incorporates feedback potential that provides students with possibilities to validate their work (Hersant & Perrin-Glorian, 2005). TDS serves as a tool both to organise and analyse teaching, as well as to hypothesise didactical situations that support students' learning. It highlights five phases of teaching – devolution, action, formulation, validation and institutionalisation – and their character, which is either didactical (performed by a teacher who is teaching directly) or adidactical (without the teacher's interference). The latter usually exists in students' work during the action, formulation and validation phases. These are the phases for students' autonomous work, in which they try to solve the problem, formulate and test hypothesis, and finally presumably formulate the new requested knowledge. Validation, if achieved by students, is beneficial if it is also recognised as correct by the other students involved, but this is the phase in which the teacher can intervene, as well. In the (didactical) devolution phase, the teacher announces the problem to the students and presents the milieu, while in the phase of institutionalisation, which is also carried out by the teacher, s/he sums up the students' solutions (new knowledge) and connects it to the official (institutional) answer to the posed problem.

The teacher's role in IBMT requires further attention. Teachers are called upon to orchestrate teaching during the inquiry process, which means they need to withdraw, to "keep their hands in the pockets", to avoid the temptation to explain to the students what to do, but also to balance the process in order to prevent students becoming frustrated when stuck. Moreover, teachers are called upon to formalise new knowledge by building on the students' ideas. The

last issue is especially recognised by Sherin (2002) as a pedagogical tension that requires the teacher to balance seemingly opposing demands between various students' ideas and productive mathematical discussion. The author notes that this demand is not easy to resolve.

The MERIA scenarios aim to provide support for teachers to orchestrate inquiry-based teaching, which they can adapt to their purposes and conditions. The scenarios are structured by the successive TDS phases, with timings that describe the lesson, especially by considering the didactical or adidactical character of the phases. In addition, the scenarios describe the assumed roles of the students and the teacher in each phase. The expected students' pre-knowledge and the assumed target knowledge are explicated, and the students' role is described by their possible strategies and learning issues for the action phase, as well as the possible realisations of the target knowledge in the formulation and validation phases as phases of the production of new knowledge. The description of the teacher's role includes an illustration of how the teacher devolves a certain phase and supports the students' activity. Some of these elements in a designed scenario can be treated as so-called didactical variables: they are the choices made by the teacher that influence learning but can be changed during a particular lesson, e.g., the number of students in a group, the anticipated time of a phase, the technology at the teacher's disposal, the size and shape of the geometrical objects in a milieu, etc. The designed scenarios do not (and cannot) anticipate all of the students' actions, but they build a hypothesised reference that allows a comparison with the classroom implementation.

When working with a task, the students' personal knowledge often develops in a way that is directly related to the context of the problem. Although it can be further developed and formalised when shared and discussed with other students, it is still different from the official (institutional) knowledge. The teacher's role in the phase of institutionalisation is didactical: s/he reformulates the students' personal knowledge relying as much as possible on their own work.

> "It is essential that the teacher challenges his students' personal knowledge by posing new problems which require knowledge they have not yet fully developed. In this way, personal knowledge is being validated. It can be validated either by the teacher, by the problem situation itself, or compared to other students, e.g. to their strategies when solving a problem. In this way, personal knowledge is transformed and becomes more formalized. This means that the knowledge becomes closer to what can be regarded as institutional knowledge." (Winsløw, 2017, p. 32).

In the process of inquiry, the teacher faces diverse student work that needs to be verified as (un)productive, situations that may have unexplored potential, overlooked student actions, or other unexpected situations. This requires the teacher to have a range of teaching skills and proficient knowledge, both mathematical and pedagogical, in order to respond to these challenges and to create opportunities for a transition between students' personal knowledge and institutional knowledge. Nonetheless, the phase of formulation provides an opportunity both for the teacher to observe students' reasoning, and for the students to practise their communication skills. Moreover, while presenting their solution, students make their stream of thought explicit, which then enables other students to engage in the discussion and provide feedback. Hence, it is not only the milieu that can be used to validate the solution, but the whole class becomes an environment that supports meaningful learning.

Bearing in mind what has been said thus far, we may state that our focus in the present article is twofold and formulate our research questions as follows:

1. How do the designed scenarios provide students with opportunities to build mathematical knowledge during classroom implementation?
2. Which elements of the scenarios related to the interaction of the teacher and the students guide the lesson to its goal?

Method

In the Erasmus+ project MERIA, several teaching scenarios were developed by teachers and researchers in mathematics education and mathematics from partner countries. They were implemented in the classroom in field trials during the school years in the period 2017–19 in schools in Croatia, Slovenia, Denmark and the Netherlands. The aim of the field trials was to test the theoretical assumptions of the design in practice in terms of their feasibility. In particular, the field trials aimed to identify the didactical actions of teachers and learning opportunities for students in order to refine the scenarios for future use. The first selection of scenarios was based on the requirement that the prescribed target knowledge formed part of curricula in all of the partner countries. The implementations of the scenarios selected for the field trials were observed by at least one teacher, usually from the same school, and the actions of the students and the teacher were documented in a report. All of the students' productions were collected. After the field trials, the next selection of proposed scenarios was made and the selected scenarios were published on the project's website. In order to discuss situations that are challenging for teachers and rich in teaching potential, we present experiences with two subsequent

scenarios, Conflict Lines – Introduction[6] and Conflict Set – Parabola[7]. Both of these scenarios are published on the project's website: the first scenario is among the selected exemplary MERIA scenarios, while the second is published in the repository. Since it has target knowledge that does not cover a curriculum topic in all of the partner countries, the second scenario was not chosen in the first group.

In Croatia, some 500 students were included in testing of the first scenario, and approximately 200 students for the second. Among all of the classroom implementations, the students' productions presented in this article were selected by teachers who participated in the field trials as the most informative. The classroom implementation of both scenarios took place in a high school with an emphasis on mathematics, science and computer science, and was performed by a (single) experienced mathematics teacher. It was implemented in the second grade in a class of 25 students. Student pre-knowledge included knowledge of a perpendicular bisector as a line perpendicular to a given segment and passing through its midpoint, as well as knowledge of the parabola as the graph of a quadratic function. During the two meetings at which the scenarios were implemented, the students worked in groups. In the formulation phase, they wrote down their hypothesis and arguments on posters to facilitate classroom discussion. All of the students' productions were collected.

As mentioned above, we used the theoretical frameworks of RME and TDS in the design of the tasks and scenarios, making theoretical assumptions on their didactical potential. We also hypothesised the teacher's role in classroom implementation. Our focus is now on exploring the implementation, on what is observed, what may happen in practice or what is missing. We focus mostly on the adidactical and feedback potential of the created learning environments hypothesised by the scenarios, when students work alone in their own inquiry and formulate their new personal knowledge. Based on the available observations, we also discuss the teacher's actions based on the MERIA scenarios in terms of which elements of the scenarios may facilitate the teacher's decisions and what is possibly missing. We describe the rich situations that the teacher may face regarding the students' formulation and validation as a basis for the teacher's institutionalisation. During the field trials, we identified the moments that were not described in detail in the scenarios but need to be considered in further institutionalisation. The implementation of the scenarios was also analysed with respect to the realisation of the TDS phases from the lesson plans. Our conclusions are based on the teachers' reflection and reports,

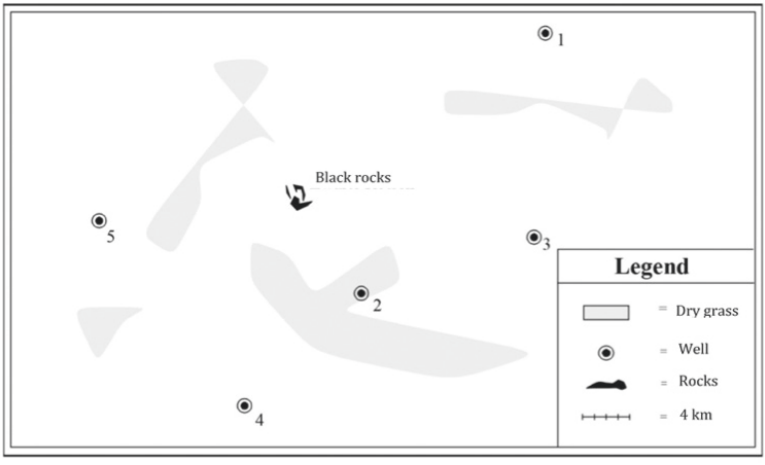
6 See more at https://meria-project.eu/activities-results/meria-teaching-scenarios.
7 See more at https://meria-project.eu/repository.

the students' work and the explicit formulation of the elements of the specific didactical situations that were hypothesised by the design (and now refined) and that could be transferred to other didactical situations.

Mathematical context

In the first scenario, Conflict Lines – Introduction, the task is intended to involve students in what is, for them, a meaningful path to the target knowledge: perpendicular bisectors are lines whose points are equidistant from a pair of points, thus creating the partition of the plane into regions, the so-called Voronoi diagram. In the second scenario, Conflict Set – Parabola, the parabola is perceived as a locus of points in the plane equidistant from a given (horizontal) line and a point. Both scenarios fall under the umbrella of *conflict sets*, that is, sets that are *equidistant* from a given set of points: in the first scenario the set of (discrete) points in the plane, and in the second scenario a line and a point.

Task 1: There are some water wells in a desert, as shown on the map. A thirsty person naturally goes to the closest well. Which well should the thirsty person go to from different points in the desert? Make a partition of the desert into areas in such a way that for all points in an area, a certain well is the closest of all of the wells.



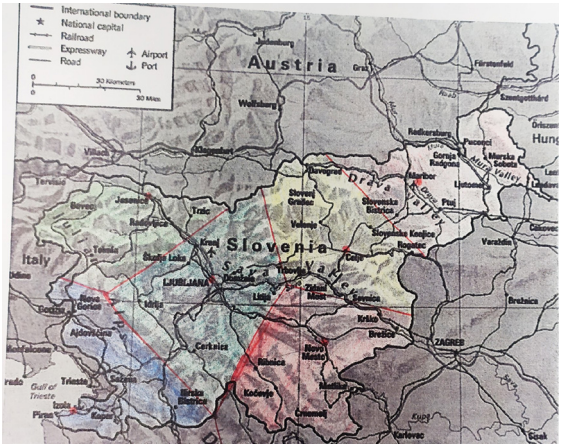
Task 2: In the coordinate plane, consider the line p given by the equation $y = 2$ and the point $A(5, 4)$. Show that points $B(1, 7)$ and $C(7, 4)$ are equally distant from the line p and point A. Find all of the points with the same property!

The first task, the water well problem, is a well-known task chosen for the present article partly because it has already been presented at a ProMath conference (Holzäpfel et al., 2016), where it was investigated as a task that offers a rich situation for students' own explorations. As stated in Holzäpfel et al. (2016), researchers' assumptions about possible strategies (heuristics) include guessing, reducing the complexity of the problem (e.g., working only with two water wells), or searching for familiar details and formulating an analogous (easier) situation (e.g., someone put one water well in another position to get a better idea of the problem). As argued by Doorman et al. (2020), the task and its possible sub-questions offer students the opportunity to work in geometry as postulated by RME, starting from their own experience and continuing to appreciate formal, precisely defined geometrical objects with their properties understood.

In the MERIA field trials, the students' pre-knowledge of perpendicular bisectors is already available from primary school. It is assumed by the scenario that in the devolution, the teacher would offer a reduction of the problem to two points. This could evoke the concept of a perpendicular bisector of a segment determined by the points, which then defines a border of the requested regions in this case. Further reduction to three points is not part of the devolution: it could evoke an additional concept, as given three points that can be considered as vertices of a triangle, there is a point that is equally distant from all of them, the centre of a circumscribed circle to a triangle. The rich context of this problem could allow further questioning about the situation of four points if there is – or when there is – a point that is equally distant from all of them. This could also be seen as a didactical variable, with the different positioning of the four points leading to a cyclic quadrilateral. Another didactical variable is the context of the task: it may be presented in a variety of other contexts, some of which could be more meaningful to the students. In the field trials during the MERIA project, one teacher from Slovenia changed the context of the task from wells in a desert to hospitals in a country that are reached by helicopters in the case of an emergency (Figure 1). This possibility was similarly suggested in Holzäpfel et al. (2016), where the context was changed to restaurants on a map.

Figure 1

Changed context of the water well task in the devolution phase



In the second task, a parabola is described as the curve (locus of points) on a plane satisfying a geometrical property, that is, as a set of points that are equidistant from a given point and a given line. The students' pre-knowledge of a parabola is that it is a graph of a quadratic function, and this knowledge was realised directly prior to the implementation of the lesson. This scenario therefore aims to support the students' linking of different viewpoints of a parabola: the geometrical viewpoint as a locus of points in a plane equidistant from a given horizontal line and a point (*parabola as a curve*), and the algebraic viewpoint as a graph of a quadratic function (*parabola as a function graph*). Both of these are curriculum topics in Croatia, although the establishment of these links is not required. The scenario also assumes a motivating activity for the students during the first devolution of the problem: one student stands in the class and the others have to form a navigation route for a robot to avoid the student and the closest wall in the class by maintaining an equal distance to them. The students' action phase is assumed to begin by checking whether two given points satisfy the condition of being equally distant from a line and a point. It is then assumed that the students will try to find more concrete points with the same property, initially probably by guessing. Since it is not a straightforward task, the teacher may suggest the use of a strategy such as considering the points lying on a certain line, considering certain symmetrical points, etc. By plotting the generated points in the coordinate system, the students may first notice that the requested equidistant set is not a line. If the shape is evoked as a parabola, the question arises as to how to prove it, that is, whether it is possible to find a quadratic relationship between the variables.

Results

Rich context with unexplored situations in Task 1

Four groups of students worked on this task and were engaged in the inquiry. They made a number of provisional drawings during the action phase. Two groups completely realised the assumed mathematical target knowledge and presented it as such in the formulation phase on a poster. The other two groups struggled with various strategies. The teacher assumed that the task, together with the described devolution based on a reduction of the problem to two points, offered enough feedback potential for the students to verify their assumptions or initial ideas. If the students failed to achieve this alone, the teacher decided to leave the case of three perpendicular bisectors for the classroom discussion as a rich unexplored situation. Here we present the final formulations of the two groups that did not get involved in exploring the case of three bisectors.

Figure 2.1
Group 1

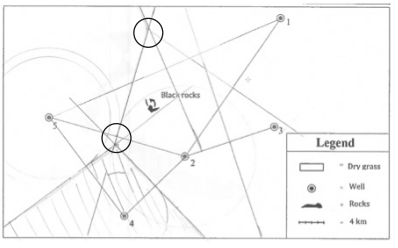
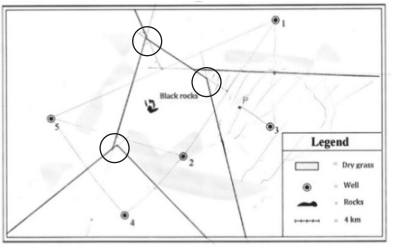


Figure 2.2
Group 2



Group 1:

The students drew circles around the wells in order to represent the idea of "being equally distant from a point" (Figure 2.1). However, they considered this strategy as unproductive and did not persist with it. They simply connected pairs of points to obtain segments and drew the perpendicular bisectors without referring further to the circles. The regions that belong to a certain well were difficult to distinguish. This group did not analyse the special case of what happens where bisectors (seem to) meet (marked by red circles).

Group 2:

The students started by connecting points and drawing perpendicular bisectors in order to determine the regions. Due to their rather imprecise work, the triples of bisectors did not meet (Figure 2.2, marked by red circles). The

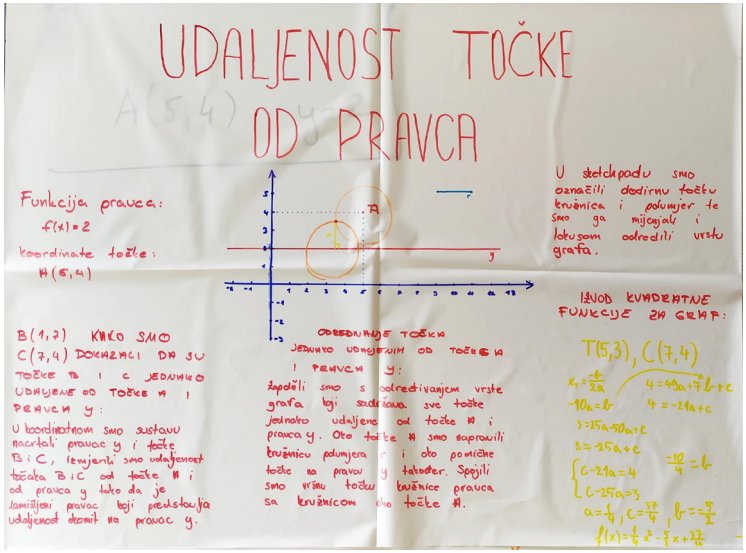
question arises as to what to do in the resulting area, that is, in the triangle, but the students left this situation unexplored.

Rich context with unexpected formulations in Task 2

In connection to Task 2, we address the students' diverse and rich solution formulations, which present a challenging task for the teacher to be evaluated during a lesson. The teacher needs to recognise productive formulations or to question misleading ones in the new personal knowledge of the students. However, the teacher usually also questions the students' formulations based on his/her own expectations regarding how far the students can or should reach, how formal the students' mathematics language or reasoning can or should be, how often s/he should interfere and/or provide a feedback, and similar. In the classroom implementation of Task 2, the students worked in four groups with the same teacher. All of the students were engaged in the inquiry. As intended by the scenario of Task 2, after the action phase, the students were asked to formulate their hypotheses and possible validations. In the validation phase, they presented their group work using posters and shared the explanations and results with their colleagues. The students were required to evaluate their own ideas and those of others. The formulations on the posters of all of the groups are presented below.

Group 1:

Figure 3
Formulation of Group 1



The first group worked with technology (Dynamic Geometry Software, DGS) from the beginning of their action phase. The students verified that points *B* and *C* were equidistant from the given point and the given straight line (Figure 3). Using technology, they tried to obtain all of the points that satisfy this property. They drew circles with the same radius around point *A* and around an arbitrary point on the line. The measurements showed that the intersection of these circles did not satisfy the property. The students therefore concluded that the point they called the "peak point" of the circle with its centre on the line should be observed (Figure 4.1). However, this "peak point" for an arbitrary point on the line (as the centre of a circle) did not even belong to the fixed circle with the centre at point *A*. The students then moved the arbitrary point along the line and thus adjusted the "peak point" to the circle with its centre at point *A* (Figure 4.2). In this way, they obtained one point of the required curve. The construction of all of the points by DGS required teacher support. After concluding that the required curve had the form of a parabola, the students used the vertex (5, 3) and point (7, 4) to set the system of equations. The solutions were the coefficients of the quadratic function $f(x) = \frac{1}{4}x^2 - \frac{5}{2}x + \frac{37}{4}$.

This group had difficulty constructing a new point equidistant to the given line and the point. They used the fact that all points on a circle are equidistant to its centre (the given point *A*), but did not know how to connect it to being equidistant to a line. They argued that it may be related to perpendicularity, which is why they invented the point called the "peak point" (Figure 4.1), that is, the intersection of the circle constructed around an arbitrary point on line $y = 2$ and a perpendicular line passing through the arbitrary point. The students experimented to obtain the requested new point (Figure 4.2).

Figure 4.1
The idea of the "peak point"

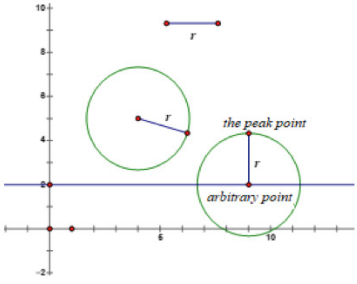
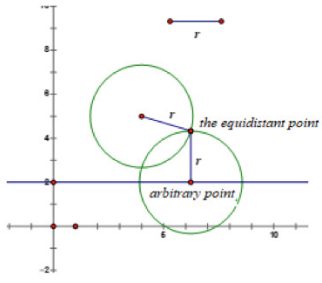
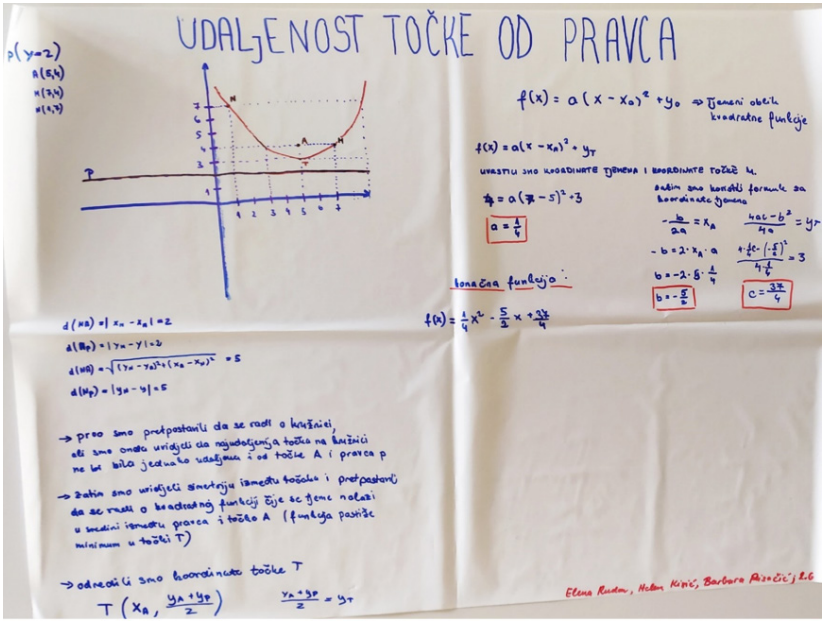


Figure 4.2
Construction of a new point



Group 2:

Figure 5
Formulation of Group 2



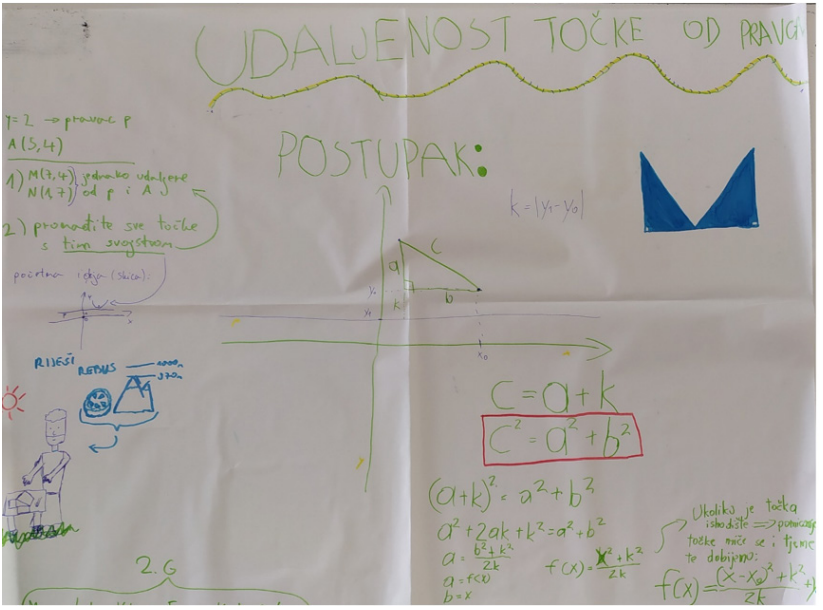
Rather than using technology, the students in this group used the distance formula to verify that points *B* and *C* satisfy the property (Figure 5). Based on the shape they formed in the first devolution (the motivating activity), they assumed it was a circle. After discussion, they rejected this assumption with the argument: *We saw that the farthest point on the circle would not be equidistant from point A and line p*. They then found several points that were equally distant from the given point and the line: the ones symmetrical with respect to the line passing through the given point and perpendicular to the given straight line. Thinking of a curve that is symmetrical, they assumed that it was a graph of a quadratic function whose vertex was halfway between the line and point *A*, and concluded that the function reached a minimum at point *T*. The students found the coordinates of point $T(x_A, \frac{y_A + y_P}{2})$ and used the vertex form of the quadratic function: $f(x) = a(x - x_A)^2 + y_T$. Substituting the coordinates of point *M* resulted in the coefficient $a = \frac{1}{4}$. Coefficients *b* and *c* were obtained by solving the system of two equations with two unknowns. They this obtained the formula of a quadratic function $f(x) = \frac{1}{4}x^2 - \frac{5}{2}x + \frac{37}{4}$.

After the teacher's instruction to prepare their arguments and conclusions for presentation, the students commenced a group discussion on how to be sure about the solution. They tried to prove that all points with coordinates $(x, ax^2 + bx + c)$ are equidistant from point $A(5, 4)$ and the straight line $y = 2$, but they failed in their attempt. In dialogue with the teacher, they concluded that they should work with points $(x, \frac{1}{4}x^2 - \frac{5}{2}x + \frac{37}{4})$. They presented this idea in the class and did the proof as homework.

Group 3:

Figure 6

Formulation of Group 3

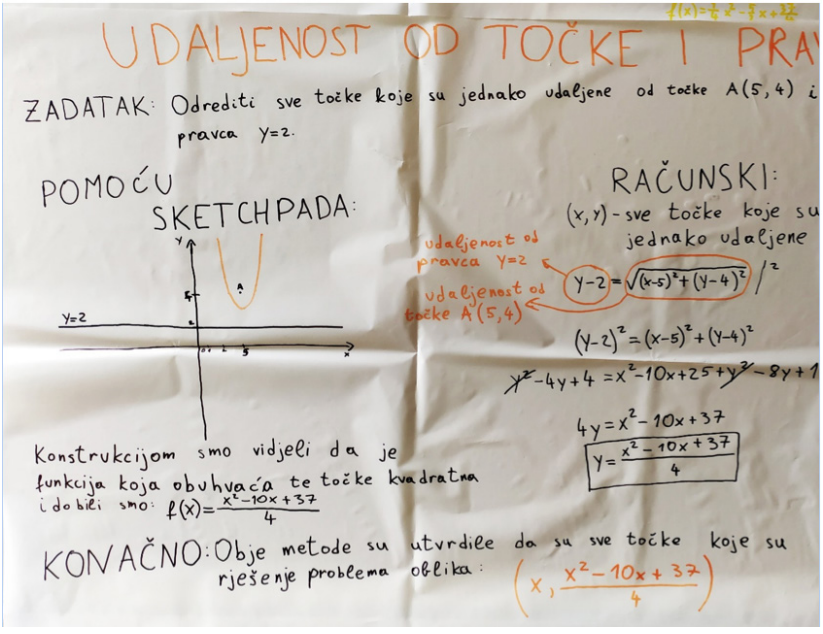


The students in this group did not use technology. They started from a point that satisfied the condition and obtained a quadratic relation of the coordinates (Figure 6). In so doing, they found the relation generally, using the translation of the coordinate system. They began with a given point as the origin and a line parallel to the x axis and k units distant from it. They denoted the point that satisfied the required condition by (b, a). The distance from that point to the origin is denoted by c. Then $c^2 = a^2 + b^2$, and according to the task $c = a + k$. From there, with a mistake in signs, they got $a = \frac{b^2 + k^2}{2k}$ (correct would be $a = \frac{b^2 - k^2}{2k}$). By substituting a = $f(x)$, $b = x$, the function becomes

$f(x) = \frac{b^2 + k^2}{2k}$ for the translated coordinate system. Finally, $f(x) = \frac{(x - x_o)^2 - k^2}{2k} + y_o$. The students stopped at this point. The correct general solution is (the difference being due to the mistake the students made). After substituting $(x_o, y_o) = (5, 4)$ and $k = 2$ into the last equation, the solution turns out to be $f(x) = \frac{x^2 - 10x + 37}{4}$, as required.

Group 4:

Figure 7
Formulation of Group 4



The students used DGS technology. They independently planned the steps of the research and independently performed the constructions (Figure 7). Using technology, they obtained points that satisfy the property and noticed that the curve has the shape of a parabola. They thus determined the rule $f(x) = \frac{x^2 - 10x + 37}{4}$. After this, they did some computation. They denoted by (x, y) all of the equidistant points and using the distance formula $y - 2 = \sqrt{(x - 5)^2 + y - 4)^2}$ resulted in $f(x) = \frac{x^2 - 10x + 37}{4}$. Finally, they concluded that both methods gave the same answer, so all of the points that satisfy the given condition have the coordinates $(x, \frac{x^2 - 10x + 37}{4})$.

Discussion

In a teaching approach such as inquiry-based teaching, according to the theoretical assumptions behind the design of the lesson, each scenario carries a certain adidactical potential that enables students to work autonomously and possibly verify their hypothesis by themselves. If students are stuck when solving a task, the teacher decides on the extent to which s/he can rely on the feedback potential offered by the task, or whether to scaffold the students in their work. In both of the classroom implementations investigated, we observed that the students were actively involved in the inquiry and found the tasks motivating and challenging to explore. We also noted that the adidactical potential of the tasks was realised, at least in creating the initial students' strategies, whereas in some groups, almost all of the students realised the expected main target knowledge, especially in the second task. The feedback potential of the first task, assumed as the possibility to directly test the properties of the points on the map, was not extensively used by the students. The two groups whose productions are presented here mostly tried to avoid such reasoning and to directly implement the conclusion from the first devolution that the perpendicular bisector provides points that are equally distant from the two points given. In the second task, the students relied much more on the feedback potential of the task and, even using technology, explored different points that enabled them to realise the pattern required.

Finally, regarding the teacher's role, the classroom implementation revealed numerous details regarding the feasibility of the scenarios. The main idea behind the scenarios is to facilitate teaching in an inquiry environment, especially for inexperienced teachers, who often express that it is not enough just to get a task. The scenarios offer the teacher some (assumed) insight into the students' thinking, which may or may not be subsequently observed in the classroom. Even more important, however, is to realise whether some crucial moments in the possible students' thinking are missing, in order to refine the scenarios. The teachers who implemented these two scenarios largely agree that the scenarios provided a solid support for their teaching. However, even in the present study, we have identified questions of a very generic nature for which the teacher needs to be prepared, and which are not emphasised in the scenarios.

In the first scenario, Conflict Lines – Introduction, a question arose concerning a particular (unexplored) rich subtask (the intersection of three perpendicular bisectors obtained by three wells). Although this question is briefly mentioned in the scenario, it could easily be overlooked by a reader

of the scenario's rather long text. If not prepared in advance, it is not easy for the teacher to recognise such rich situations during the lesson. This situation provides an opportunity for the teacher to scaffold the students and to establish links with their prior knowledge.

In the second scenario, Conflict Set – Parabola, in the first group, the teacher was challenged to grasp the informal student communication that emerged with the students' creative productive strategy. The teacher asked the students to explain the meaning and deepen the idea of the *peak point*. Although the students failed to express themselves using more formal mathematical language, the teacher encouraged them to persist. It is up to the teacher whether to question the students and to determine the level of mathematical formality that s/he is satisfied with.

In the second group, the students initially assumed that the required set of points was a circle, and the teacher did not intervene. It is very common for a teacher to have an initial urge to correct the student's reasoning; however, the teacher decided to remain withdrawn. It was up to the students to realise that their hypothesis was incorrect. The teacher's preparation allowed her to consider that the milieu ensured enough potential for the students to test and validate their hypothesis. After the students had rejected the idea of a circle, they assumed that the solution was a parabola and used the data to determine the equation. Finally, they proved that all points on the parabola, that is, on a graph of the quadratic function $f(x) = \frac{1}{4}x^2 - \frac{5}{2}x + \frac{37}{4}$, are equidistant from the given point and line.

In the third group, the teacher noticed that there was a mistake in the calculation that would not influence the idea behind the solution. The strategy itself was very innovative and unexpected, as the group translated the coordinate system in order to simplify calculations, although they did not express their idea in this way. At first, the students' sketch was also misleading, giving the impression that they had used similar triangles with no clear rationale. The students' reasoning was very informal, and the calculation was incorrect and incompletely provided. This strategy was not anticipated in the scenario, so the teacher needed to recognise that the students' idea was different than the official solution due to the mistake they had made. The teacher had an opportunity to discuss the innovative idea, understanding that the mistake did not have a major impact.

In the fourth group, from the perspective of teacher, the solution was the most complete. After plotting some points that satisfied the required property, the students devised the hypothesis that the solution was a parabola and calculated its coefficients. Seeing it was still a hypothesis, the students autonomously decided that they needed some kind of validation. They therefore verified the solution

algebraically starting from the property of equal distance and ended up with the same equation. The students called this general argument "the computational approach". Still, the teacher did not interfere with the work of this group, although the precise mathematical statement involved logical equivalence in formulation. Regarding the mathematical target knowledge, the students needed to discover that the point lies on the conflict set *if and only if* its coordinates satisfy a certain equation. Students usually find a way to show one direction or the other, but not both. In this case, Group 2 proved one direction and Group 4 the other. Thus, the formulation phase in this task provides an opportunity for the teacher to build on the students' work in the institutionalisation phase, and to discuss the logical equivalence in the statement as a subtle interplay between algebra and geometry: a set of points that is the *conflict set* of a point and a line is given by quadratic dependence (the parabola as a graph of a quadratic function) and, vice-versa, all points of a parabola as a graph of a quadratic function has the property of the *conflict set*. The teacher can encourage students to compare the proofs given by Group 2 and Group 4, and thus to notice equivalence.

At the end of a lesson designed in this way, after the different strategies are presented by the groups, the teacher needs to decide what can be validated and institutionalised for all of the students. The final discussion allows the student to confront her/his new personal knowledge with the knowledge of other students, to improve it, and finally, with the help of a teacher, to connect it with the official knowledge.

Conclusion

Teaching is a complex task in an open inquiry environment, requiring the teacher to have a range of proficiency skills. It is more challenging than addressing rote techniques by direct teaching. It seems that not only mathematical and pedagogical knowledge and skills, but also beliefs influence teachers' behaviour (Rott, 2020). Beliefs are especially important for IBMT because teachers may incorporate their previous experiences and belief systems concerning mathematics into their teaching, which can be a constraint when introducing IBMT into practice. However, pursuing the IBMT approach, we still assume that "the goal of teaching is for the students to acquire a certain established and culturally recognized knowledge, which they will be able subsequently to use without the teacher's help" (Hersant & Perrin-Glorian, 2005, p.113). Teaching scenarios designed for the purpose of IBMT within the Erasmus+ project MERIA offer a solid ground for teachers to orchestrate such teaching. The scenarios have been designed based on two theoretical frameworks, RME and

TDS, and the experience that the members of the project team brought from different institutions. The presented teaching sequence on *Conflict Sets* has not yet been thoroughly discussed within the scope of the project, and the analysis presented in this paper completes that work. In this article, we discussed several student productions as evidence of the students' newly constructed knowledge. These productions present the rich and inspiring situations that the teachers confronted and exploited to build new official knowledge. In addition to the didactical situations hypothesised by the teaching scenarios, we described the outcomes of their classroom implementation, which show that the theoretical ideas behind the design of the scenarios can work in practice, and that the scenarios helped the teacher to navigate the inquiry process. We observed the diverse student reactions, but also noted that it was feasible for the teacher to perceive a variety of strategies and proceed with the lesson by using them. However, we also identified situations that were not emphasised by the scenarios. These situations indicate the numerous creative moments when teachers make decisions, create opportunities and challenge the situation in order to support a productive exchange of mathematical ideas, and as such could be of interest to any practising teacher. In future investigations, we aim to consider whether we can refine our investigation into the characteristics of tasks for secondary school mathematics in addition to their adidactical potential, such as *linkage, deepening and research potential*, as assumed for the purpose of university mathematics education in Gravesen et al. (2017). Such studies are expected to contribute to a better understanding of how to support teachers in the crucial and creative moments when they try to recognise and use opportunities for moving between students' discoveries and intended target knowledge.

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